High frequency multimode radiation from ducts with flow

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Multimode sound radiation from hard-walled semi-infinite ducts with uniform subsonic flow is investigated theoretically. An analytic expression, valid in the high frequency limit, is derived for the multimode directivity function in the forward arc for a general family of mode distribution functions. The multimode directivity depends on the amplitude and directivity function of each individual mode. The amplitude of each mode is expressed as a function of cut-off ratio for a uniform distribution of incoherent monopoles, a uniform distribution of incoherent axial dipoles and for equal power per mode. The modes' directivity functions are obtained analytically by applying a Lorentz transformation to the zero flow solution. The analytic formula for the multimode directivity with flow is derived assuming total transmission of power at the open-end of the duct. This formula is compared to the exact numerical result for an unflanged duct, computed utilizing a Wiener–Hopf solution. The agreement is shown to be excellent.

I. Introduction

Background: Duct power estimation from limited far-field data

Various common noise sources radiate sound into finite length ducts containing a uniform mean flow, from which the sound escapes into the far field via radiation from an unbaffled open end. Examples are exhaust mufflers, large exhaust stacks, and aircraft turbofan engines. Often one wishes to determine the sound power radiated from the duct opening, either as an index of insertion loss in order to assess silencer performance, or as a means of quantifying and ranking the total noise output for predicting community annoyance. The sound power may, in principle, be determined by integrating the normal component of sound intensity over a surface enclosing the duct exit at a large distance from the duct where the flow is quiescent. However, sometimes not all measurement locations required to perform the integration are easily accessible, as in the case of large exhaust stacks which may be tens of meters high. In this example, the only measurements which are easy to make are close to the ground, corresponding to the rear-arc or backward-radiated sound radiated at angles approaching 180° to the duct axis. It is clear that a method of inferring the radiated power, at any frequency, from a small number of far-field mean square pressure measurements would be extremely useful to the noise control engineer.

Scope of investigation

This paper presents a theoretical and numerical study of the non-dimensional directivity factor $Q(ka, \theta)$ for multimode sound radiation from the open end of a semi-infinite duct, in the presence of a uniform mean flow. This paper extends earlier work by the second author of this paper in which the effects of flow were neglected.[1] The directivity factor relates the mean square far-field pressure, at any polar angle θ to the duct axis including the rear arc, and at any flow Mach number, to the net sound power transmitted along the duct. The non-dimensional frequency ka equals $2\pi fa/c$, where f is the frequency, a is the duct radius, and c is

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the sound speed. For calculating the modal radiation in the presence of flow (assumed everywhere the same) we Lorentz transform the exact theoretical zero-flow expressions for modal sound radiation from a semi-infinite, hard-walled, unflanged circular duct as presented by Homicz and Lordi.[2] This single-mode analysis is applied in what follows to a particular family of mode amplitude weighting functions, which includes the following three special cases: (a) excitation of incident modes by incoherent monopoles uniformly distributed over a duct cross section; (b) excitation by incoherent axial dipoles uniformly distributed over a duct cross section; (c) equal incident in-duct power per mode above cutoff. In each case, the individual modes are incoherently excited, and the contribution of evanescent modes is neglected. For these three incident-mode weighting models in turn, we present graphs of $Q(ka, \theta)$ as a function of θ for a number of representative Mach numbers (positive and negative representing exhaust and inlet conditions) in both the forward and rear arc. The paper concludes by deriving high-frequency asymptotic expressions for $Q(ka, \theta)$ for the three source models that explicitly includes the effect of uniform mean flow. Because scattering at the edge of the duct opening is neglected in the high-frequency model, this analysis is limited to the forward arc.

II. Far field pressure directivity with flow

Consider a semi-infinite hard-walled cylindrical duct of radius a immersed in a uniform flow of Mach number M, which is assumed to be identical inside and outside the duct. Let (r, ϕ, z) be the cylindrical polar coordinate system illustrated in figure 1. At a single frequency ω, the acoustic field in a semi-infinite duct containing a uniform mean flow may be expressed in the form

$$p(r, \phi, z, \omega) = \sum_{(m,n)\in\mathcal{O}} A_{mn}\Psi_{mn}(r, \phi)e^{-jk_{z,mn}z} , \tag{1}$$

where here the mode order (m, n) is restricted to the set of cut-on modes, denoted by $\mathcal{O}$. In equation (1) A_{mn} is the modal pressure amplitude, Ψ_{mn} is the ortho-normal mode shape function of the duct defined such that

$$\frac{1}{\pi a^2}\int_0^{2\pi}\int_0^a \Psi_{mn}(r, \phi)\Psi^*_{m'n'}(r, \phi)r \; \mathrm{d}r \, \mathrm{d}\phi = \delta_{mn,m'n'} , \tag{2}$$

where $\delta_{mn,m'n'}$ is the Kronecker delta function, and $k_{z,mn}$ denotes the modal axial wavenumber given by

$$k_{z,mn} = \frac{(\alpha_{mn} - M)k}{\beta^2} \quad \text{where} \quad \alpha_{mn} = \sqrt{1 - \left(\frac{\kappa_{mn}}{k}\right)^2 \beta^2} . \tag{3}$$

Also, $k = \omega/c$ denotes the free field acoustic wavenumber, $\beta = \sqrt{1 - M^2}$, α_{mn} is the modal cut-off ratio, and $\kappa_{mn} = j_{mn}/a$ is the mode transverse wavenumber where j_{mn} equals the value of the nth turning point of the Bessel function of order m. The modal cut-off ratio α_{mn} is in the range $0 \le \alpha_{mn} \le 1$ for propagating modes, where $\alpha_{mn} = 0$ corresponding to a mode that is just cut-on. For evanescent modes, α_{mn} has a non-zero imaginary part. In this paper, the evanescent modes are neglected in the modal summation which gives the in-duct pressure field. Thus, in equation (1), the sum is restricted to the set of so-called cut-on modes.

Having prescribed the in-duct pressure field (1), to express the mean square pressure $\overline{p^2}$ in the far field, it is convenient to use the spherical polar coordinate system (R, ϕ, θ) as shown in figure 2. It is assumed that the modes are incoherent, i.e.

$$E\{A_{mn}A^*_{m'n'}\} = 0 \quad \text{if} \quad m \neq m' \quad \text{or} \quad n \neq n', \tag{4}$$

where $E\{\,\cdot\,\}$ denotes the expected value. Making this assumption greatly simplifies the expression for $\overline{p^2}$ and ensures that the radiated field is axi-symmetric, i.e. there is no dependence on azimuthal angle ϕ. Thus,

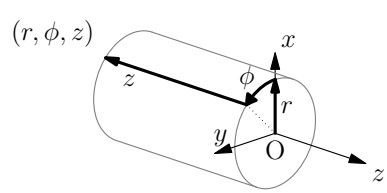


Figure 1: Cylindrical polar coordinate system (r, ϕ, z)

for incoherent modes, the far field mean square pressure $\overline{p_f^2}$ can be expressed as

$$\overline{p_f^2}(R,\theta) = \frac{1}{2}\left(\frac{a}{R}\right)^2 \sum_{(m,n)\in\mathcal{O}} |H_{mn}(ka,\theta)|^2 E\{|A_{mn}|^2\}, \tag{5}$$

where $H_{mn}(ka,\theta)$ is the in-duct to far field transfer function previously defined for zero-flow by Joseph *et al.*[1]

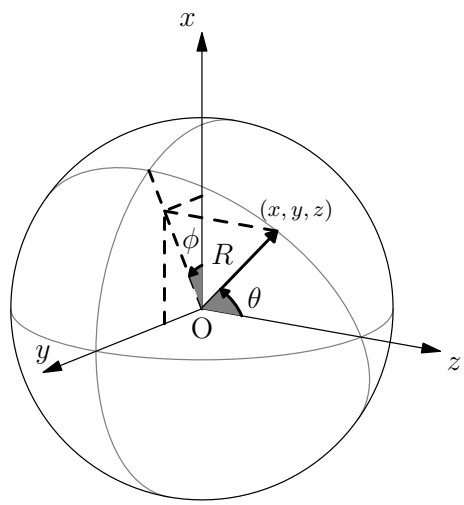


Figure 2: Spherical coordinate system (R,ϕ,θ)

II.A. Transfer functions for modal radiation from ducts with flow

In this section, the transfer function H_{mn} for ducts immersed in a uniform flow is related to the transfer function for modal radiation in the absence of flow, which will be denoted with the superscript '0'.

Exhaust duct

For an exhaust duct ($M \geq 0$), in which sound is radiated in the direction of the flow, the pressure is continuous everywhere. Therefore, the in-duct to far-field transfer function with flow may be obtained by direct Lorentz transformation of the zero-flow far field pressure solution, using for example the procedure presented by Chapman,[3] which gives

$$H_{mn}(ka,\theta) = \frac{\beta}{\sqrt{1-M^2\sin^2\theta}} H^0_{mn}(ka/\beta, \tan^{-1}(\beta\tan\theta)) \tag{6}$$

where H^0_{mn} is the transfer function for modal sound radiation in the absence of flow. Equation (6) was originally derived by Homicz and Lordi.[2] In Ref. [2], an exact expression for this zero-flow transfer function is provided utilizing the Wiener–Hopf technique. However, an approximate transfer function also may be utilized such as the expression for a flanged duct provided by a Rayleigh integral, as described by Tyler and Sofrin.[4] This is naturally restricted to polar angles in the forward arc.

Inlet duct

For an inlet duct ($M \leq 0$), in which sound is radiated against the direction of the flow, the pressure is singular at the open end of the duct (on the duct rim). Thus, the Lorentz transformation cannot be applied directly to the zero-flow pressure solution. However, the velocity potential is continuous everywhere, and the Lorentz transformation of the zero-flow velocity potential solution can be taken, to give an expression for the velocity potential with flow. Then, the far field pressure is found from the velocity potential solution via the acoutsic momentum equation. Following this procedure, this gives

$$H_{mn}(ka,\theta) = \frac{\beta}{1-M\alpha_{mn}} \frac{\sqrt{1-M^2\sin^2\theta} - M\cos\theta}{1-M^2\sin^2\theta} H^0_{mn}(ka/\beta, \tan^{-1}(\beta\tan\theta)) \,. \tag{7}$$

Equation (7) was also originally derived by Homicz and Lordi,[2] but Equation (7) differs slightly from the equation in Ref. [2]. Our result is recovered by replacing $\cos\psi$ in the expression in Ref. [2] by $\cos\tilde{\psi}$.

II.B. Modal amplitudes

In this paper the behaviour of the far-field mean square pressure is investigated for a general family of mode distribution functions $E\{|A_{mn}|^2\}$. Special cases include an incoherent monopole or axial dipole source uniformly distributed over the cross-section of the duct, and a random source distribution which generates an equal power per mode sound field.

The mode amplitude distribution function may be obtained by generalizing the approach taken by Joseph and Morfey[1] for zero flow. The result is,

$$E\{|A_{mn}|^2\} = A^2_{\mu,\nu,\gamma}(\alpha_{mn}) \equiv P^2_{\mu,\nu,\gamma}\left(\frac{M-\alpha_{mn}}{1-M^2}\right)^{2\mu}\left(\frac{1-\alpha_{mn}M}{1-M^2}\right)^{2\nu}\frac{1}{\alpha^{\gamma}_{mn}}, \tag{8}$$

where $P_{\mu,\nu,\gamma}$ is a measure of source strength and (μ,ν,γ) denotes a trio of indices which characterizes the source model. Special cases of equation (8) are listed in table 1.

	Monopole	Dipole	Equal power per mode
$P^2_{\mu,\nu,\gamma}$	$(\rho_0 c_0)^2\overline{Q^2_s}/2$	$\overline{F^2_s}/2$	$2\rho_0 c_0 W_0/(\pi a^2)$
(μ,ν,γ)	(0,1,2)	(1,0,2)	(0,1,1)

Table 1: modal amplitudes for three model of sources in a semi-infinite cylindrical duct with flow

The formula given in equation (8) is more general than the source models derived previously by Joseph *et al*,[5] but at the expense of having to include an additional exponent, γ, which allows for the equal power per mode model to be included in the general family of mode distribution functions.

III. Multimode directivity factor $Q(ka,\theta)$ with flow

In this paper the non-dimensional multimode directivity function defined by

$$Q(ka,\theta) \equiv \frac{4\pi R^2}{\rho_0 c_0 E\{W_f\}}\overline{p^2_f}(R,\theta), \tag{9}$$

where W_f denotes the radiated acoustic power, is investigated. Following the definition of generalized acoustic intensity given, for example, by Morfey,[6] it can be shown that the radiated power is given by

$$E\{W_f\} = \frac{\pi R^2}{\rho_0 c_0}\int_0^{\pi}\overline{p^2_f}(R,\theta)F(\theta)\sin\theta d\theta\ d\theta, \tag{10}$$

where $F(\theta)$ is defined as

$$F(\theta) = \frac{\beta^4\sqrt{1-M^2\sin^2\theta}}{2\left(\sqrt{1-M^2\sin^2\theta}-M\cos\theta\right)^2}. \tag{11}$$

Note that for zero-flow, $F(\theta) = 1/2$, and equation (10) reduces to the classical result for radiated sound power.

Combining equations (9) and (11) shows that Q satisfies the normalization condition

$$\int_0^{\pi} Q(ka,\theta)F(\theta)\sin\theta\ d\theta = 1. \tag{12}$$

Computation of equation (9) using equation (5) is in general a computationally intensive procedure for large ka and M. For example, at $ka = 50$, there are approximately 650 cut-on modes at $M = 0$. This number increases as $(1-M^2)^{-1/2}$ as M increases. Therefore, there is a need for a simple analytic formula that provides a good approximation to $Q(ka,\theta)$ in the high frequency limit, for the family of sound sources described in section II.B. The derivation of this analytic formula is presented in the following section.

IV. Analytic expression for $Q(ka, \theta)$ in the high ka limit

In this section, analytic formulas are derived for the multimode directivity function Q with flow for the general family of source distributions given in section II.B.

First consider the sound power $E\{W_{mn}\}$ transmitted by a single mode wave travelling along the duct with flow. Following Morfey,[6]

$$E\{W_{mn}\} = \frac{\pi a^2}{2\rho_0 c_0} A^2_{\mu\nu\gamma}(\alpha_{mn}) \frac{\alpha_{mn}(1-M^2)^2}{(1-\alpha_{mn}M)^2} \,. \tag{13}$$

Sound power variation with θ

Following equation (10), the far field sound power $dW_f(\theta)$ radiated between angles θ and $\theta + d\theta$ is given by

$$E\{dW_f(\theta)\} = \frac{4\pi R^2}{\rho_0 c_0} \overline{p_f^2}(R, \phi, \theta) F(\theta) \sin\theta d\theta \,. \tag{14}$$

Relation between far-field power and in-duct power

For the case where the flow speed is everywhere the same, and assuming that no energy is lost at the open-end of the duct, the power radiated between the range of angles θ and $\theta + d\theta$ is equal to the power transported between this range of angles inside the duct, i.e.

$$dW_f(\theta) = dW(\theta) \,. \tag{15}$$

In-duct power

The in-duct sound power transmitted along the duct between propagation angles θ and $\theta + d\theta$ is the sum of the modal power transmitted by each mode whose propagation angle θ_{mn} is between θ and $\theta + d\theta$, i.e.

$$dW(\theta) = \sum_{(m,n)\in\mathcal{O}_\theta} W_{mn} \,, \tag{16}$$

where

$$\mathcal{O}_\theta = \Big\{(m,n) \in \mathcal{O} \;\; | \;\; \theta \le \theta_{mn} \le \theta + d\theta\Big\} \,. \tag{17}$$

Combining equations (13) and (16) gives,

$$E\{dW(\theta)\} = \frac{\pi a^2}{2\rho_0 c_0} \sum_{(m,n)\in\mathcal{O}_\theta} \frac{\beta^4 \alpha_{mn}}{(1-\alpha_{mn}M)^2} A^2_{\mu\nu\gamma}(\alpha_{mn}). \tag{18}$$

The propagation angle θ_{mn} can be related to α_{mn} and M using results presented by Rice *et al.*,[7] i.e.

$$\cos\theta_{mn} = \frac{\alpha_{mn}\beta}{\sqrt{1-M^2\alpha_{mn}^2}} \,. \tag{19}$$

For each mode $(m,n) \in \mathcal{O}_\theta$, assuming that $d\theta$ is small, then assume that $\theta_{mn} \approx \theta$. Therefore, equation (19) yields

$$\alpha_{mn} = \frac{\cos\theta_{mn}}{\sqrt{1-M^2\sin^2\theta_{mn}}} \approx \frac{\cos\theta}{\sqrt{1-M^2\sin^2\theta}} = \alpha(\theta) \,. \tag{20}$$

Then, substituting this into equation (18) gives,

$$E\{dW(\theta)\} = \frac{\pi a^2}{2\rho_0 c_0} dN(\theta) \frac{\beta^4 \alpha(\theta)}{(1-\alpha(\theta)M)^2} A^2_{\mu\nu\gamma}(\alpha(\theta)) \,, \tag{21}$$

where $dN(\theta)$ denotes the number of modes with propagation angle between θ and $\theta + d\theta$.

The derivation of $dN(\theta)$ is in the Appendix. It is shown that, as $ka \to \infty$,

$$dN(\theta) = \frac{(ka)^2}{2} \frac{\sin\theta\cos\theta}{(1-M^2\sin^2\theta)^2} d\theta \,. \tag{22}$$

Using this asymptotic expression for $dN(\theta)$ leads to the following analytic expression for the in duct sound power transmitted at polar angle θ:

$$E\{dW(\theta)\} = \frac{(ka)^2\pi a^2}{4\rho_0 c_0}\frac{\sin\theta\cos\theta}{(1-M^2\sin^2\theta)^2}\frac{\beta^4\alpha(\theta)}{(1-\alpha(\theta)M)^2}A^2_{\mu\nu\gamma}(\alpha(\theta))d\theta\,. \tag{23}$$

IV.A. Multimode far field directivity

Equating the in-duct sound power (23) to the radiated power expression (14), and solving for the mean square pressure gives,

$$\overline{p_f^2}(R,\theta) = \left(\frac{a}{R}\right)^2\frac{(ka)^2}{8}\frac{\cos^2\theta}{(1-M^2\sin^2\theta)^2}A^2_{\mu,\nu,\gamma}(\alpha(\theta)), \tag{24}$$

where we have used the expressions for $F(\theta)$ and $\alpha(\theta)$ given in equations (11) and (20) to simplify the result.

The mean square pressure for the special cases: (a) a uniform distribution of incoherent monopoles; (b) a uniform distribution of incoherent axial dipoles; (c) equal power per mode, are given by

$$\begin{aligned}
&\text{(a)}\quad \overline{p_f^2}(R,\theta) = (\rho_0 c_0)^2\overline{Q_s^2}\left(\frac{a}{R}\right)^2\frac{(ka)^2}{16}\frac{(\sqrt{1-M^2\sin^2\theta}-M\cos\theta)^2}{\beta^4(1-M^2\sin^2\theta)^2},\\
&\text{(b)}\quad \overline{p_f^2}(R,\theta) = \overline{F_s^2}\left(\frac{a}{R}\right)^2\frac{(ka)^2}{16}\frac{(M\sqrt{1-M^2\sin^2\theta}-\cos\theta)^2}{\beta^4(1-M^2\sin^2\theta)^2},\\
&\text{(c)}\quad \overline{p_f^2}(R,\theta) = \frac{W_0}{4\pi R^2}(ka)^2\rho_0 c_0\frac{(\sqrt{1-M^2\sin^2\theta}-M\cos\theta)^2\cos\theta}{\beta^4(1-M^2\sin^2\theta)^{5/2}}.
\end{aligned} \tag{25}$$

IV.B. Non-dimensional directivity factor Q

Now utilizing the generalised directivity expression given by equation (24), the radiated sound power defined by equation (10) is computed. The expected value of the source power can be obtained by integrating the elementary sound power contributions $E\{dW(\theta)\}$ between 0 and $\pi/2$. The contributions from angles between $\pi/2$ and π are neglected. Thus, from equation (23),

$$E\{W\} = \frac{\pi a^2}{\rho_0 c_0}\frac{(ka)^2}{4}\int_0^{\pi/2}\frac{\sin\theta\cos\theta}{(1-M^2\sin^2\theta)^2}\frac{\beta^4\alpha(\theta)}{(1-\alpha(\theta)M)^2}A^2_{\mu\nu\gamma}(\alpha(\theta))\;d\theta \quad \text{as} \quad ka\to\infty\,. \tag{26}$$

The integration can be carried out analytically for the three source distributions presented previously, which gives

$$(a)\quad E\{W\} = \frac{\overline{Q_s^2}}{2}\frac{(ka)^2}{4\beta^2}\pi a^2\rho_0 c_0, \tag{27}$$

$$(b)\quad E\{W\} = \frac{\overline{F_s^2}}{2}\frac{(ka)^2}{4\beta^2}\pi a^2\frac{1}{\rho_0 c_0}\frac{2\beta^2\log(1-M)+2M+M^2-M^3-M^4}{M^3}, \tag{28}$$

$$(c)\quad E\{W\} = W_0\frac{(ka)^2}{4\beta^2}. \tag{29}$$

Note that since $(ka)^2/4\beta^2$ is the analytic expression for the total number of propagating modes in the duct, equation (29) based on an equal power per mode model, has a simple interpretation.

Substituting the analytical expressions for pressure directivity (24) and power (26), into the definition of $Q(ka,\theta)$ (9), leads to

$$Q(ka,\theta) = \frac{2A^2_{\mu,\nu,\gamma}(\alpha(\theta))\cos^2\theta/(1-M^2\sin^2\theta)^2}{\beta^4\int_0^{\pi/2}(\sin\theta\cos\theta)\alpha(\theta)A^2_{\mu\nu\gamma}(\alpha(\theta))/[(1-M^2\cos^2\theta)^2(1-\alpha(\theta)M)^2]\;d\theta}\,. \tag{30}$$

In the high frequency limit $ka \to \infty$, the expression for Q can be evaluated analytically for each special case of idealized source distribution. This gives

$$(a) \quad Q_{0,1,2} = 2\frac{(\sqrt{1-M^2\sin^2\theta} - M\cos\theta)^2}{\beta^2(1-M^2\sin^2\theta)^2}, \tag{31}$$

$$(b) \quad Q_{1,0,2} = \frac{2}{\beta^2}\frac{M^3}{2M+M^2-M^3-M^4+2\beta^2\log(1-M)}\frac{(M\sqrt{1-M^2\sin^2\theta} - \cos\theta)^2}{(1-M^2\sin^2\theta)^2}, \tag{32}$$

$$(c) \quad Q_{0,1,1} = 4\frac{(\sqrt{1-M^2\sin^2\theta} - M\cos\theta)^2}{\beta^2(1-M^2\sin^2\theta)^{5/2}}\cos\theta. \tag{33}$$

Note that as $M \to 0$,

$$Q_{0,1,2} \to 2, \quad Q_{1,0,2} \to 6\cos^2\theta, \quad Q_{0,1,1} \to 4\cos\theta. \tag{34}$$

which are the directivity factors derived by Joseph and Morfey[1] for zero flow.

V. Results

The directivity function Q is shown plotted in figures 3, 4 and 5 for $ka = 50$, and a uniform distribution of incoherent monopoles, a uniform distribution of incoherent axial dipoles, and equal power per mode, respectively. Each figure is composed of five polar plots corresponding to Mach number $M = -0.9, -0.3, 0, 0.3, 0.9$. In each polar plot the grey line shows the exact directivity function with flow computed from equation (9) using the single-mode directivity functions derived by Homicz and Lordi.[2] The round circles correspond to the high frequency approximation (30), which is valid in the forward arc. Agreement between the two solutions is excellent up to about 80°.

For a uniform distribution of incoherent monopoles (figure 3), the directivity is omnidirectional in the forward arc for $M = 0$. For an inlet duct ($M < 0$), as the flow speed increases, Q increases approximately uniformly over the forward arc. A different behaviour is observed for an exhaust duct ($M > 0$). In this case, as the flow speed increases, Q decreases at polar angles close to the duct axis and increases at angles further away from the duct axis.

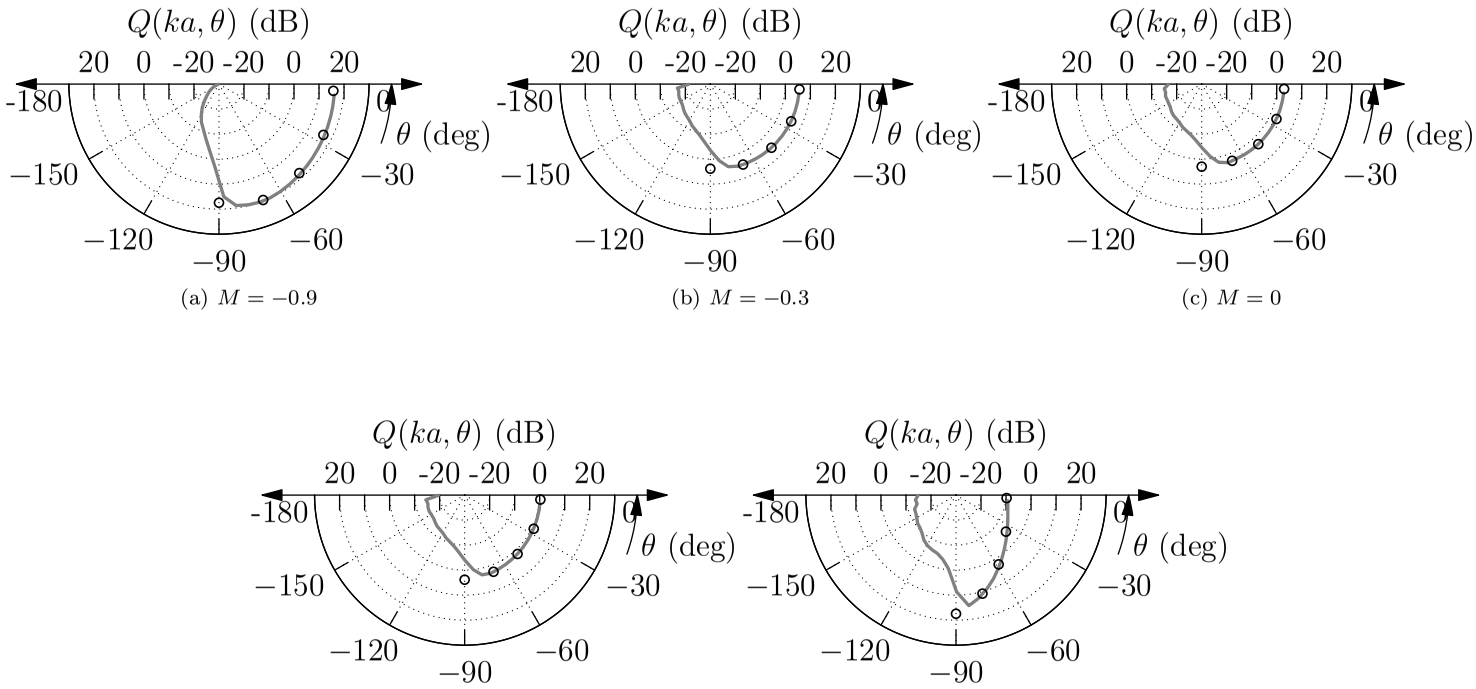
For a uniform distribution of incoherent axial dipoles (figure 4), the radiation is not as uniform over the forward arc for zero flow compared with the distribution of incoherent monopoles. However, for an inlet duct ($M < 0$), as the flow speed increases, the directivity pattern tends to become uniform in the forward arc. Contrastingly, for an exhaust duct ($M > 0$), the directivity is zero at certain angles, and tends to radiate more efficiently away from the duct axis as the flow speed increases. From equation (32), the position of the zeros is given by

$$\cos\theta = \frac{M}{\sqrt{1-M^2}}. \tag{35}$$

Finally, for equal power per mode (figure 5), the behaviour is similar to that for a uniform distribution of incoherent monopoles.

VI. Conclusion

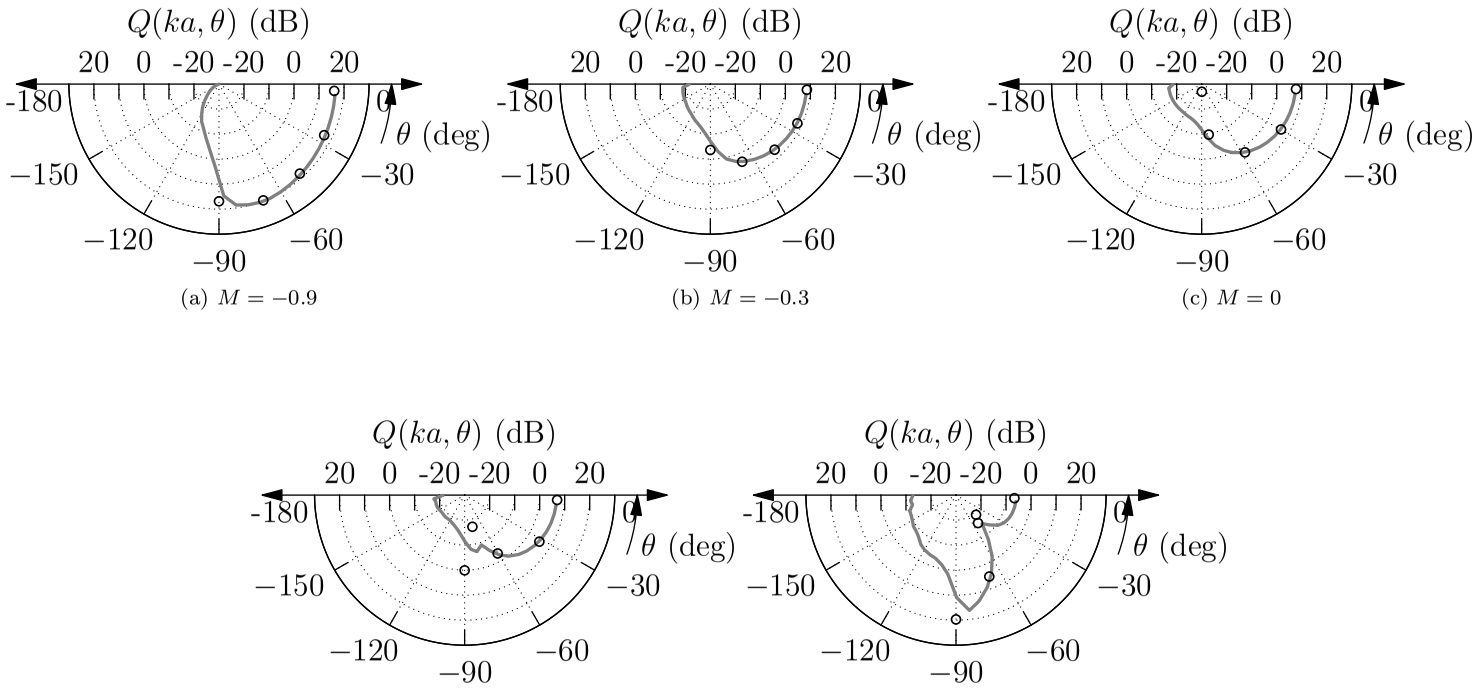
This paper has investigated the directivity function of multimode sound radiation from an unflanged circular hard-walled duct in which the flow speed is everywhere the same. Both inlet and exhaust cases are investigated. This problem is of importance in the understanding of the radiation of fan broadband noise from aero engine ducts. The behaviour of the multimode radiation is considered for a general family of mode distribution function. Special cases are incoherent monopoles uniformly distributed over the duct cross section, a uniform distribution of axial dipole sources, and the case of equal power per mode. A non-dimensional directivity function $Q(ka, \theta)$ is defined normalized with respect to the total radiated acoustic power. Analytic expressions for Q are derived for these three source models that are valid in the high frequency limit and in the forward arc. The dependence on the radiation angle θ and flow Mach number are revealed explicitly.



(a) $M = -0.9$ (b) $M = -0.3$ (c) $M = 0$

(d) $M = 0.3$ (e) $M = 0.9$

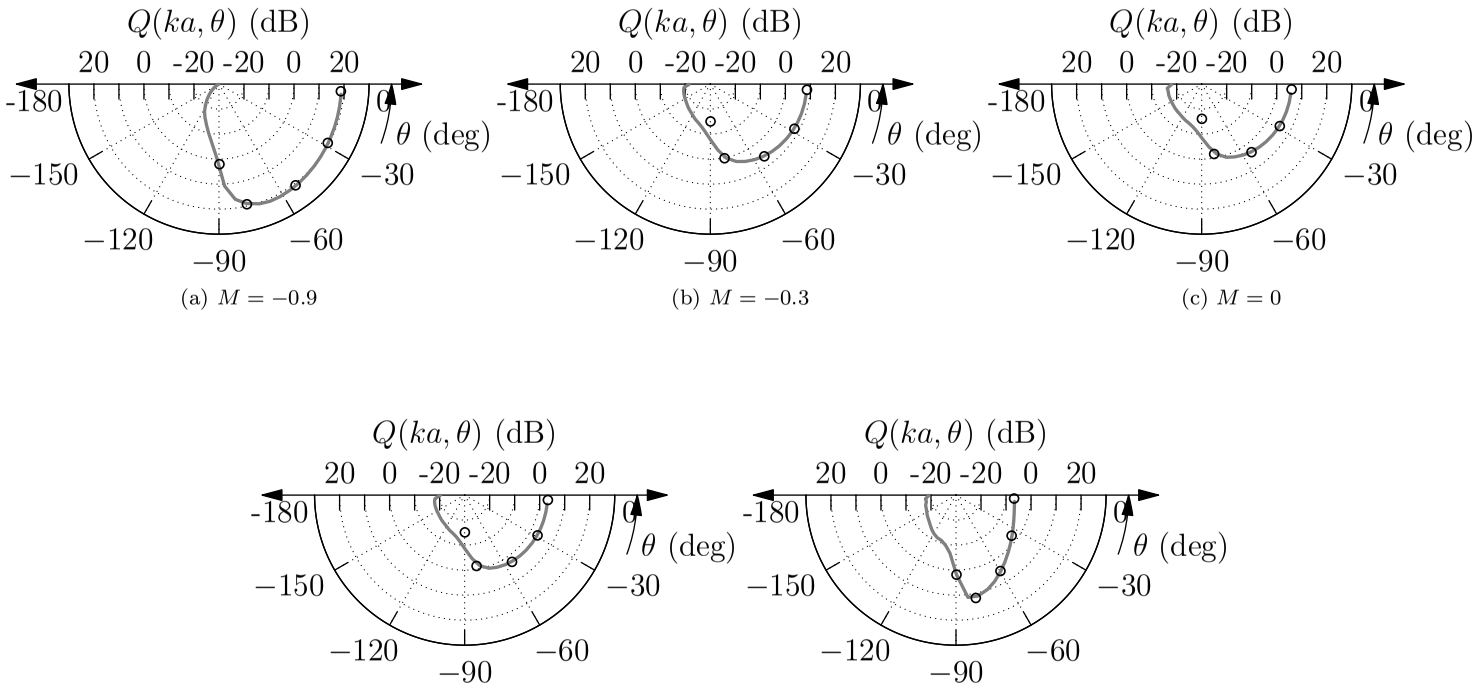
Figure 3: Directivity factor from a cylindrical duct in a uniform flow, for a uniform distribution of incoherent monopoles, for different values of Mach number M and $ka = 50$



(a) $M = -0.9$ (b) $M = -0.3$ (c) $M = 0$

(d) $M = 0.3$ (e) $M = 0.9$

Figure 4: Directivity factor from a cylindrical duct in a uniform flow, for a uniform distribution of incoherent axial dipoles, for different values of Mach number M and $ka = 50$



(a) $M = -0.9$ (b) $M = -0.3$ (c) $M = 0$

(d) $M = 0.3$ (e) $M = 0.9$

Figure 5: Directivity factor from a cylindrical duct in a uniform flow, for an equal power per mode model, for different values of Mach number M and $ka = 50$

A. Derivation of $dN(\theta)$

In this appendix, we derive the number of modes with propagation angle between θ and $\theta + d\theta$, denoted by $dN(\theta)$. It corresponds to the number of elements in $\mathcal{O}_\theta$, which has been defined in equation (17). To estimate $dN(\theta)$ we introduce the function $G(\theta)$ defined as the number of modes (m, n) such that $\theta_{mn} \leq \theta$. By definition of $G(\theta)$,

$$dN(\theta) = G(\theta + d\theta) - G(\theta). \tag{36}$$

Since $d\theta$ is small, the above expression can be expressed in terms of the derivative of $G(\theta)$, denoted by $G'(\theta)$, i.e.

$$dN(\theta) = G'(\theta) d\theta \,. \tag{37}$$

In the above expression, $G'(\theta)$ can be interpreted as a modal density.

The function $G(\theta)$ can be estimated as follows. First observe that these two statements are equivalent:

$$0 \leq \theta_{mn} \leq \theta \leq \frac{\pi}{2} \qquad \text{and} \qquad k \sin\theta_{mn} \leq k \sin\theta \,. \tag{38}$$

Then, from equation (19), one can relate $\sin\theta_{mn}$ to $\sin\theta$ as follows. Given that

$$k \sin\theta_{mn} = \frac{\beta \kappa_{mn}}{\sqrt{1 - M^2 \alpha_{mn}^2}} \,, \tag{39}$$

and by analogy, define $\kappa(\theta)$ as

$$\kappa(\theta) = \frac{k}{\beta} \sin\theta \sqrt{1 - M^2 \alpha(\theta)^2}. \tag{40}$$

Substituting the expression for $\alpha(\theta)$, given in equation (20), $\kappa(\theta)$ can be expressed as

$$\kappa(\theta) = \frac{k \sin(\theta)}{\sqrt{1 - M^2 \sin^2\theta}} \,. \tag{41}$$

For $0 \leq \theta \leq \pi/2$, the cosine and sine functions are decreasing and increasing, respectively. Therefore, it is straightforward to show that $\alpha(\theta)$ is decreasing with θ. Thus,

$$0 \leq \theta_{mn} \leq \theta \leq \frac{\pi}{2} \qquad \text{and} \qquad \sqrt{1 - M^2 \alpha_{mn}^2} \leq \sqrt{1 - M^2 \alpha^2(\theta)} \,, \tag{42}$$

are equivalent statements. Combining equations (38) and (42), then

$$0 \leq \theta_{mn} \leq \theta \leq \frac{\pi}{2} \qquad \text{and} \qquad \kappa_{mn} \leq \kappa(\theta) \,, \tag{43}$$

are also equivalent statements. This explicitly shows that modes satisfying $\theta_{mn} \leq \theta$ are identical to those satisfying $\kappa_{mn} \leq \kappa(\theta)$.

The advantage of utilizing κ_{mn} is that it is independent of the presence of flow. Therefore, the results available for zero flow concerning κ_{mn} can be readily applied to the flow case.

Let $\mathcal{M}(\kappa)$ denote the number of cut-on modes (m, n) such that $\kappa_{mn} \leq \kappa$. From equation (43),

$$G(\theta) = \mathcal{M}(\kappa(\theta)). \tag{44}$$

Accurate analytic approximations for $\mathcal{M}(\kappa)$ have been derived previously. For example, following Roe[8] and Rice,[9]

$$\mathcal{M}(\kappa) = \frac{(\kappa a)^2}{4} + \frac{\kappa a}{2} \approx \frac{(\kappa a)^2}{4} \,, \tag{45}$$

which is a good approximation when $\kappa a \gg 1$ (i.e. for well cut-on modes).

Combining equations (44) and (45), $G(\theta)$ can be expressed as

$$G(\theta) = \frac{(ka)^2}{4} \frac{\sin^2\theta}{1 - M^2 \sin^2\theta} \,. \tag{46}$$

Differentiating this gives

$$G'(\theta) = \frac{(ka)^2}{2} \frac{\sin\theta\cos\theta}{(1 - M^2\sin^2\theta)^2} \,. \tag{47}$$

Thence, substituting equation (47) into equation (37), the number of cut-on modes which radiate between angle θ and $\theta + d\theta$ is given by

$$dN(\theta) = \frac{(ka)^2}{2} \frac{\sin\theta\cos\theta}{(1 - M^2\sin^2\theta)^2} d\theta \,. \tag{48}$$

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